

The maximum product of sizes of cross-intersecting families

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Abstract

We say that a set A *t-intersects* a set B if A and B have at least t common elements. Two families \mathcal{A} and \mathcal{B} of sets are said to be *cross-t-intersecting* if each set in \mathcal{A} *t-intersects* each set in \mathcal{B} . A subfamily \mathcal{S} of a family \mathcal{F} is called a *t-star of \mathcal{F}* if the sets in \mathcal{S} have t common elements. Let $l(\mathcal{F}, t)$ denote the size of a largest *t-star* of \mathcal{F} . We call \mathcal{F} a $(\leq r)$ -family if each set in \mathcal{F} has at most r elements. We determine a function $c : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that the following holds. If \mathcal{A} is a subfamily of a $(\leq r)$ -family \mathcal{F} with $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$, \mathcal{B} is a subfamily of a $(\leq s)$ -family \mathcal{G} with $l(\mathcal{G}, t) \geq c(r, s, t)l(\mathcal{G}, t+1)$, and \mathcal{A} and \mathcal{B} are cross-*t-intersecting*, then $|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t)l(\mathcal{G}, t)$. Some known results follow from this, and we identify several natural classes of families for which the bound is attained.

1 Introduction

Unless otherwise stated, we shall use small letters such as x to denote non-negative integers or set elements or functions, capital letters such as X to denote sets, and calligraphic letters such as \mathcal{F} to denote *families* (that is, sets whose members are sets themselves). The set $\{1, 2, \dots\}$ of all positive integers is denoted by \mathbb{N} . For any $m, n \in \mathbb{N}$ with $m < n$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$. We abbreviate $[1, n]$ to $[n]$. It is to be assumed that arbitrary sets and families are *finite*. We call a set A an *r-element set*, or simply an *r-set*, if its size $|A|$ is r . For a set X , the *power set of X* (that is, the family of all subsets of X) is denoted by 2^X , and the family of all *r-element subsets of X* is denoted by $\binom{X}{r}$.

We say that a set A *t-intersects* a set B if A and B contain at least t common elements. A family \mathcal{A} of sets is said to be *t-intersecting* if every two sets in \mathcal{A} *t-intersect*. A 1-intersecting family is also simply called an *intersecting family*.

For a family \mathcal{F} and a set T , we denote the family $\{F \in \mathcal{F} : T \subseteq F\}$ by $\mathcal{F}(T)$. We call $\mathcal{F}(T)$ a *t-star of \mathcal{F}* if $|T| = t$. A *t-star* of a family is the simplest example of a *t-intersecting subfamily*. We denote the size of a largest *t-star* of \mathcal{F} by $l(\mathcal{F}, t)$.

We denote the set of largest t -stars of \mathcal{F} by $L(\mathcal{F}, t)$. We say that \mathcal{F} has the t -star property if at least one t -star of \mathcal{F} is a largest t -intersecting subfamily of \mathcal{F} .

One of the most popular endeavours in extremal set theory is that of determining the size or the structure of a largest t -intersecting subfamily of a given family \mathcal{F} . This originated in [23], which features the classical Erdős-Ko-Rado (EKR) Theorem. The EKR Theorem says that, for $1 \leq t \leq r$, there exists an integer $n_0(r, t)$ such that, for every $n \geq n_0(r, t)$, the size of a largest t -intersecting subfamily of $\binom{[n]}{r}$ is $\binom{n-t}{r-t}$, meaning that $\binom{[n]}{r}$ has the t -star property. It was also shown in [23] that the smallest possible value of $n_0(r, 1)$ is $2r$, and two of the various proofs of this fact (see [38, 36, 19]) are particularly short and beautiful: Katona's [36], introducing the elegant cycle method, and Daykin's [19], using the Kruskal-Katona Theorem [39, 37]. If $n/2 < r < n$, then $\binom{[n]}{r}$ itself is intersecting. A sequence of results [23, 26, 52, 28, 1] culminated in the complete solution of the problem for t -intersecting subfamilies of $\binom{[n]}{r}$. The solution confirmed a conjecture of Frankl [26] and particularly tells us that $\binom{[n]}{r}$ has the t -star property if and only if $n \geq (t+1)(r-t+1)$ [26, 52]. The same t -intersection problem for $2^{[n]}$ was solved by Katona [38]. These are among the most prominent results in extremal set theory. The EKR Theorem inspired a wealth of results that establish how large a system of sets can be under certain intersection conditions; see [21, 27, 25, 34, 35, 11].

Two families \mathcal{A} and \mathcal{B} are said to be *cross- t -intersecting* if each set in \mathcal{A} t -intersects each set in \mathcal{B} . More generally, k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross- t -intersecting* if for every i and j in $[k]$ with $i \neq j$, each set in \mathcal{A}_i t -intersects each set in \mathcal{A}_j . Cross-1-intersecting families are also simply called *cross-intersecting families*.

For t -intersecting subfamilies of a given family \mathcal{F} , the natural question to ask is how large they can be. For cross- t -intersecting families, two natural parameters arise: the sum and the product of sizes of the cross- t -intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of k -tuples (A_1, \dots, A_k) such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross- t -intersecting subfamilies (not necessarily distinct or non-empty) of a given family \mathcal{F} . The paper [15] analyses this problem in general, particularly reducing it to the problem of maximising the size of a t -intersecting subfamily of \mathcal{F} for k sufficiently large. Solutions have been obtained for various families (see [15]).

Wang and Zhang [51] solved the maximum sum problem for an important class of families that particularly includes $\binom{[n]}{r}$, using a striking combination of the method in [6, 7, 8, 16, 9] and an important lemma that is found in [3, 17] and is referred to as the *no-homomorphism lemma*. The solution for $\binom{[n]}{r}$ with $t = 1$ had been obtained by Hilton [32] and is the first result of this kind. For $2^{[n]}$, the maximum sum problem was solved [15, Theorems 3.10, 4.1] via the result in [51], and the maximum product problem was settled in [43] for the case where $k = 2$ or $n + t$ is even (see [15, Section 5.2], which features a conjecture for the case where $k > 2$ and $n + t$ is odd).

In this paper, we address the maximum product problem for the more general setting where each \mathcal{A}_i is a subfamily of a family \mathcal{F}_i . This has been considered for a

few special families [48, 44, 33, 13, 5], and, as we explain below, in many cases it is enough to solve the problem for $k = 2$ (see Lemma 2.5).

The maximum product problem for $\binom{[n]}{r}$ was first addressed by Pyber [48], who proved that, for $r, s, n \in \mathbb{N}$ such that either $r = s \leq n/2$ or $r < s$ and $n \geq 2s + r - 2$, if $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that \mathcal{A} and \mathcal{B} are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{r-1} \binom{n-1}{s-1}$. Subsequently, Matsumoto and Tokushige [44] proved this for $r \leq s \leq n/2$ (see also [4]). For cross- t -intersecting subfamilies, we have the following.

Theorem 1.1 ([13]) *For $1 \leq t \leq r \leq s$, there exists an integer $n_0(r, s, t)$ such that, for every $n \geq n_0(r, s, t)$, if $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{B} \subseteq \binom{[n]}{s}$, and \mathcal{A} and \mathcal{B} are cross- t -intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{r-t} \binom{n-t}{s-t}$, and equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{r} : T \subseteq A\}$ and $\mathcal{B} = \{B \in \binom{[n]}{s} : T \subseteq B\}$ for some $T \in \binom{[n]}{t}$.*

Hirschorn made a Frankl-type conjecture [33, Conjecture 4] for any r, s, t and n . A value of $n_0(r, s, t)$ that is close to best possible is established in [14]. The special case $r = s$ is treated in [49, 50, 29], which establish values of $n_0(r, r, t)$ that are also nearly optimal.

Let $c : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that, for $r, s, t \in \mathbb{N}$, $c(r, s, t) = \max\{r \binom{s}{t}, s \binom{r}{t}\} + 1$ if $t \leq \min\{r, s\}$, and $c(r, s, t) = 1$ otherwise. Clearly, $c(r, s, t) = r \binom{s}{t} + 1$ for $t \leq r \leq s$.

The following is our main result, proved in Section 3.

Theorem 1.2 *If $r, s, t \in \mathbb{N}$, \mathcal{F} is a $(\leq r)$ -family with $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$, \mathcal{G} is a $(\leq s)$ -family with $l(\mathcal{G}, t) \geq c(r, s, t)l(\mathcal{G}, t+1)$, and \mathcal{A} and \mathcal{B} are cross- t -intersecting families such that $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{G}$, then*

$$|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t)l(\mathcal{G}, t),$$

and equality holds if and only if $\mathcal{A} = \mathcal{F}(T) \in L(\mathcal{F}, t)$ and $\mathcal{B} = \mathcal{G}(T) \in L(\mathcal{G}, t)$ for some t -set T .

As we show in Section 4, this solves the problem for many natural families with a sufficiently large parameter depending on r, s and t . For example, Theorem 1.2 yields Theorem 1.1 by taking n large enough so that $\binom{n-t}{r-t} \geq c(r, s, t) \binom{n-t-1}{r-t-1}$; see Section 4.1.

For $r, s, t \in \mathbb{N}$, let $\chi(r, s, t)$ be the smallest non-negative real number a such that $|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t)l(\mathcal{G}, t)$ for every $\mathcal{A}, \mathcal{B}, \mathcal{F}$ and \mathcal{G} such that \mathcal{F} is a $(\leq r)$ -family with $l(\mathcal{F}, t) \geq a \cdot l(\mathcal{F}, t+1)$, \mathcal{G} is a $(\leq s)$ -family with $l(\mathcal{G}, t) \geq a \cdot l(\mathcal{G}, t+1)$, $\mathcal{A} \subseteq \mathcal{F}$, $\mathcal{B} \subseteq \mathcal{G}$, and \mathcal{A} and \mathcal{B} are cross- t -intersecting.

Problem 1.3 *What is the value of $\chi(r, s, t)$?*

By Theorem 1.2, $\chi(r, s, t) \leq c(r, s, t)$.

In Theorem 1.2, the case $\mathcal{F} = \mathcal{G}$ is of particular importance. First of all, it implies that \mathcal{F} has the t -star property if $l(\mathcal{F}, t) \geq c(r, r, t)l(\mathcal{F}, t+1)$.

Theorem 1.4 *If $1 \leq t \leq r$ and \mathcal{A} is a t -intersecting subfamily of a $(\leq r)$ -family \mathcal{F} with $l(\mathcal{F}, t) \geq c(r, r, t)l(\mathcal{F}, t+1)$, then*

$$|\mathcal{A}| \leq l(\mathcal{F}, t),$$

and equality holds if and only if $\mathcal{A} \in \mathcal{L}(\mathcal{F}, t)$.

Proof. Let $\mathcal{G} = \mathcal{F}$ and $\mathcal{B} = \mathcal{A}$. Since \mathcal{A} is t -intersecting, \mathcal{A} and \mathcal{B} are cross- t -intersecting. By Theorem 1.2, the result follows. \square

Also note that in Theorem 1.2 with $\mathcal{F} = \mathcal{G}$, the bound is attained by taking $\mathcal{A} = \mathcal{B} \in \mathcal{L}(\mathcal{F}, t)$; a generalization of this fact is given by Proposition 2.3. As we show in Example 2.1, for $\mathcal{F} \neq \mathcal{G}$, it may be that the bound is not attained, and we may also have \mathcal{A} and \mathcal{B} for which no t -set T satisfies $|\mathcal{A}||\mathcal{B}| \leq |\mathcal{F}(T)||\mathcal{G}(T)|$, no matter how large $\frac{l(\mathcal{F}, t)}{l(\mathcal{F}, t+1)}$ and $\frac{l(\mathcal{G}, t)}{l(\mathcal{G}, t+1)}$ are required to be. In view of this, we will now introduce further definitions. We will also generalise Theorem 1.2 to a result for k cross- t -intersecting families.

2 The cross- t -star property

If $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross- t -intersecting families, then we say that the tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ is *cross- t -intersecting*.

Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be families. We say that $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ is *below* $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ if $\mathcal{A}_i \subseteq \mathcal{F}_i$ for each $i \in [k]$. We say that $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the

- (a) *cross- t -star property* if $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k l(\mathcal{F}_i, t)$ for each cross- t -intersecting tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$.
- (b) *strict cross- t -star property* if, for each cross- t -intersecting tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$, $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k l(\mathcal{F}_i, t)$, and the inequality is strict if there exists no t -set T such that, for each $i \in [k]$, $\mathcal{A}_i = \mathcal{F}_i(T)$.
- (c) *strong cross- t -star property* if, for some t -set T , $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k |\mathcal{F}_i(T)|$ for each cross- t -intersecting tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$.
- (d) *extrastrong cross- t -star property* if there exists a t -set T such that, for each cross- t -intersecting tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$, $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k |\mathcal{F}_i(T)|$, and equality holds only if there exists a t -set T' such that, for each $i \in [k]$, $\mathcal{A}_i = \mathcal{F}_i(T')$.

Note that each of (b)–(d) implies (a), and (d) implies (a)–(c). As we demonstrate in Example 2.1, it may be that (b) holds, (c) does not hold, and hence (d) does not hold; clearly, this is the case only if $\prod_{i=1}^k |\mathcal{A}_i| < \prod_{i=1}^k l(\mathcal{F}_i, t)$ for each cross- t -intersecting tuple $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$.

Example 2.1 Let $r_1, \dots, r_k, t \in \mathbb{N}$ with $k \geq 2$ and $t < r_1 \leq \dots \leq r_k$. Let $T_1, \dots, T_k, A_{1,1}, \dots, A_{1,q_1}, \dots, A_{k,1}, \dots, A_{k,q_k}$ be pairwise disjoint sets such that, for each $i \in [k]$, $|T_i| = t$ and $|A_{i,1}| = \dots = |A_{i,q_i}| = r_i - t$. Let R_1, \dots, R_{k-1} be sets such that $|R_i| = r_i$ for each $i \in [k-1]$, $R_1 \subseteq \dots \subseteq R_{k-1}$, and $R_{k-1} \cap \bigcup_{i=1}^k \bigcup_{j=1}^{q_i} (T_i \cup A_{i,j}) = \emptyset$ (that is, no set R_m intersects a set $T_i \cup A_{i,j}$). For each $i \in [k-1]$, let $\mathcal{F}_i = \{T_i \cup A_{i,1}, \dots, T_i \cup A_{i,q_i}, R_i\}$. Let $\mathcal{F}_k = \{T \cup A_{k,j} : T \in \binom{R_1}{t}, j \in [q_k]\}$. For each $i \in [k]$, each set in \mathcal{F}_i is of size r_i , and clearly $l(\mathcal{F}_i, t) = q_i$. For every $i, j \in [k]$ with $i < j$, a set A in \mathcal{F}_i t -intersects a set B in \mathcal{F}_j if and only if $A = R_i$ and either $j < k$ and $B = R_j$ or $j = k$ and $B \in \mathcal{F}_j$. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a cross- t -intersecting tuple below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ such that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are non-empty (so that $\prod_{i=1}^k |\mathcal{A}_i| \neq 0$). Then $\mathcal{A}_i = \{R_i\}$ for each $i \in [k-1]$. Thus $\prod_{i=1}^k |\mathcal{A}_i| \leq |\mathcal{F}_k|$, and equality holds if and only if $(\mathcal{A}_1, \dots, \mathcal{A}_k) = (\{R_1\}, \dots, \{R_{k-1}\}, \mathcal{F}_k)$. Therefore, if $\prod_{i=1}^{k-1} q_i > \binom{r_1}{t}$, then $\prod_{i=1}^k |\mathcal{A}_i| < \prod_{i=1}^k l(\mathcal{F}_i, t)$ (since $\prod_{i=1}^k |\mathcal{A}_i| \leq |\mathcal{F}_k| = \binom{r_1}{t} q_k$ and $\prod_{i=1}^k l(\mathcal{F}_i, t) = \prod_{i=1}^k q_i$), meaning that $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property. Now let T be a t -set such that $\prod_{i=1}^k |\mathcal{F}_i(T)| \neq 0$. Then $T \subseteq R_1$ and $\mathcal{F}_i(T) = \{R_i\}$ for each $i \in [k-1]$. Thus $\prod_{i=1}^k |\mathcal{F}_i(T)| = |\mathcal{F}_k(T)| < \prod_{i=1}^k |\mathcal{A}_i|$ if $(\mathcal{A}_1, \dots, \mathcal{A}_k) = (\{R_1\}, \dots, \{R_{k-1}\}, \mathcal{F}_k)$. Therefore, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ does not have the strong cross- t -star property.

Remark 2.2 By Example 2.1, for $1 \leq t < r \leq s$, there is no real number a such that $(\mathcal{F}, \mathcal{G})$ has the strong cross- t -star property for every $(\leq r)$ -family \mathcal{F} with $l(\mathcal{F}, t) \geq a \cdot l(\mathcal{F}, t+1)$ and every $(\leq s)$ -family \mathcal{G} with $l(\mathcal{G}, t) \geq a \cdot l(\mathcal{G}, t+1)$.

Of particular importance is the case $\mathcal{F}_1 = \dots = \mathcal{F}_k$.

Proposition 2.3 (i) If $\mathcal{F}_1 = \dots = \mathcal{F}_k$ and $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the cross- t -star property, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strong cross- t -star property.

(ii) If $\mathcal{F}_1 = \dots = \mathcal{F}_k$ and $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.

Proof. Suppose $\mathcal{F}_1 = \dots = \mathcal{F}_k$. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a cross- t -intersecting tuple below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$. Let T be a t -set such that $|\mathcal{F}_1(T)| = l(\mathcal{F}_1, t)$. Since $\mathcal{F}_1 = \dots = \mathcal{F}_k$, $|\mathcal{F}_i(T)| = l(\mathcal{F}_i, t)$ for each $i \in [k]$. If $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the cross- t -star property, then $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k |\mathcal{F}_i(T)|$. If $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property and $\prod_{i=1}^k |\mathcal{A}_i| = \prod_{i=1}^k |\mathcal{F}_i(T)|$, then there exists a t -set T' such that, for each $i \in [k]$, $\mathcal{A}_i = \mathcal{F}_i(T')$. \square

Proposition 2.4 The tuple $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property if it has the strict cross- t -star property and there exists a t -set T such that, for each $i \in [k]$, $\mathcal{F}_i(T) \in L(\mathcal{F}_i, t)$.

Proof. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a cross- t -intersecting tuple below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$. Under the given conditions, $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k l(\mathcal{F}_i, t) = \prod_{i=1}^k |\mathcal{F}_i(T)|$, and $\prod_{i=1}^k |\mathcal{A}_i| = \prod_{i=1}^k l(\mathcal{F}_i, t)$ only if there exists a t -set T' such that, for each $i \in [k]$, $\mathcal{A}_i = \mathcal{F}_i(T')$. \square

By Theorem 1.2, $(\mathcal{F}, \mathcal{G})$ has the cross- t -star property if $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$ and $l(\mathcal{G}, t) \geq c(r, s, t)l(\mathcal{G}, t+1)$.

The following generalisation of [15, Lemma 5.1] follows immediately from [15, Lemma 5.2] and particularly tells us that the cross- t -star property is guaranteed for k families if it is guaranteed for every two of them.

Lemma 2.5 *If $2 \leq p \leq k$ and $\mathcal{F}_1, \dots, \mathcal{F}_k$ are families such that $(\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_p})$ has the cross- t -star property for each p -element subset $\{i_1, \dots, i_p\}$ of $[k]$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the cross- t -star property.*

For example, Theorem 1.2 yields the following generalization.

Theorem 2.6 *If $1 \leq t \leq r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, \mathcal{F}_i is a $(\leq r_i)$ -family with $l(\mathcal{F}_i, t) \geq c(r_{k-1}, r_k, t)l(\mathcal{F}_i, t+1)$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property.*

We now start working towards the proofs of Theorems 1.2 and 2.6. Then, in Section 4, we apply the results above to several important families.

3 Proof of the main result

If a set T t -intersects each set in a family \mathcal{A} , then we call T a t -transversal of \mathcal{A} .

Lemma 3.1 *If T is a t -transversal of a subfamily \mathcal{A} of a family \mathcal{F} , then*

$$|\mathcal{A}| \leq \binom{|T|}{t} l(\mathcal{F}, t).$$

Proof. Let $\mathcal{T} = \binom{T}{t}$. Since $|A \cap T| \geq t$ for all $A \in \mathcal{A}$, we have

$$|\mathcal{A}| = \left| \bigcup_{I \in \mathcal{T}} \mathcal{A}(I) \right| \leq \sum_{I \in \mathcal{T}} |\mathcal{A}(I)| \leq \sum_{I \in \mathcal{T}} |\mathcal{F}(I)| \leq \sum_{I \in \mathcal{T}} l(\mathcal{F}, t) = |\mathcal{T}| l(\mathcal{F}, t),$$

and hence the result. \square

Lemma 3.2 *If T is a t -transversal of a subfamily \mathcal{A} of a family \mathcal{F} , X is a set of size t , $\mathcal{A} \subseteq \mathcal{F}(X)$, and $X \not\subseteq T$, then*

$$|\mathcal{A}| \leq |T \setminus X| l(\mathcal{F}, t+1).$$

Proof. The result is trivial if $\mathcal{A} = \emptyset$. Suppose $\mathcal{A} \neq \emptyset$. For each $A \in \mathcal{A}$, we have

$$t \leq |A \cap T| = |A \cap (T \cap X)| + |A \cap (T \setminus X)| = |T \cap X| + |A \cap (T \setminus X)| \leq t-1 + |A \cap (T \setminus X)|,$$

and hence $|A \cap (T \setminus X)| \geq 1$. Together with $\mathcal{A} \subseteq \mathcal{F}(X)$, this gives us

$$\begin{aligned} \mathcal{A} &\subseteq \{F \in \mathcal{F} : X \subseteq F, |F \cap (T \setminus X)| \geq 1\} \\ &= \{F \in \mathcal{F} : X \cup \{y\} \subseteq F \text{ for some } y \in T \setminus X\} = \bigcup_{y \in T \setminus X} \mathcal{F}(X \cup \{y\}). \end{aligned}$$

Thus $|\mathcal{A}| \leq \sum_{y \in T \setminus X} |\mathcal{F}(X \cup \{y\})| \leq \sum_{y \in T \setminus X} l(\mathcal{F}, t+1) = |T \setminus X| l(\mathcal{F}, t+1)$. \square

We can now prove Theorem 1.2. We will call a t -intersecting family \mathcal{A} *trivial* if the sets in \mathcal{A} have at least t common elements.

Proof of Theorem 1.2. Suppose $|F| < t$ for each $F \in \mathcal{F}$. Then $L(\mathcal{F}, t) = \{\emptyset\}$, and hence $l(\mathcal{F}, t) = 0 = l(\mathcal{F}, t+1)$. Also, $|F \cap G| < t$ for each $F \in \mathcal{F}$ and each $G \in \mathcal{G}$. Thus, since \mathcal{A} and \mathcal{B} are cross- t -intersecting, one of \mathcal{A} and \mathcal{B} is empty, and hence $|\mathcal{A}||\mathcal{B}| = 0 = l(\mathcal{F}, t)l(\mathcal{G}, t)$. Similarly, $|\mathcal{A}||\mathcal{B}| = 0 = l(\mathcal{F}, t)l(\mathcal{G}, t)$ if $|G| < t$ for each $G \in \mathcal{G}$.

Now suppose that each of \mathcal{F} and \mathcal{G} has a set of size at least t . Then $r \geq t$, $s \geq t$, $l(\mathcal{F}, t) \geq 1$ and $l(\mathcal{G}, t) \geq 1$.

If one of \mathcal{A} and \mathcal{B} is empty, then $|\mathcal{A}||\mathcal{B}| = 0 < l(\mathcal{F}, t)l(\mathcal{G}, t)$.

Suppose $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. Since \mathcal{A} and \mathcal{B} are cross- t -intersecting, each set in \mathcal{A} is a t -transversal of \mathcal{B} , and each set in \mathcal{B} is a t -transversal of \mathcal{A} .

Case 1: \mathcal{A} is not a trivial t -intersecting family, and \mathcal{B} is not a trivial t -intersecting family. Let $D \in \mathcal{B}$. For each $X \in \binom{D}{t}$, let $\mathcal{A}_X = \mathcal{A}(X)$. Since $|A \cap D| \geq t$ for each $A \in \mathcal{A}$, $\mathcal{A} = \bigcup_{X \in \binom{D}{t}} \mathcal{A}_X$.

Consider any $X \in \binom{D}{t}$. Since \mathcal{B} is not a trivial t -intersecting family, there exists $B \in \mathcal{B}$ such that $X \not\subseteq B$. Since B is a t -transversal of \mathcal{A}_X , $|\mathcal{A}_X| \leq |B \setminus X| l(\mathcal{F}, t+1) \leq s \cdot l(\mathcal{F}, t+1)$ by Lemma 3.2.

Therefore, we have

$$|\mathcal{A}| = \left| \bigcup_{X \in \binom{D}{t}} \mathcal{A}_X \right| \leq \sum_{X \in \binom{D}{t}} |\mathcal{A}_X| \leq \sum_{X \in \binom{D}{t}} s l(\mathcal{F}, t+1) = s \binom{|D|}{t} l(\mathcal{F}, t+1),$$

and hence $|\mathcal{A}| \leq s \binom{s}{t} l(\mathcal{F}, t+1)$. By a similar argument, $|\mathcal{B}| \leq r \binom{r}{t} l(\mathcal{G}, t+1)$. It follows that

$$|\mathcal{A}||\mathcal{B}| \leq s \binom{s}{t} l(\mathcal{F}, t+1) r \binom{r}{t} l(\mathcal{G}, t+1) \leq r \binom{r}{t} s \binom{s}{t} \frac{l(\mathcal{F}, t)}{c(r, s, t)} \frac{l(\mathcal{G}, t)}{c(r, s, t)} < l(\mathcal{F}, t) l(\mathcal{G}, t).$$

Case 2: \mathcal{A} is a trivial t -intersecting family, and \mathcal{B} is not a trivial t -intersecting family. We have $\mathcal{A} \subseteq \mathcal{F}(X)$ for some set X of size t . Since \mathcal{B} is not a trivial t -intersecting family, there exists $B \in \mathcal{B}$ such that $X \not\subseteq B$. By Lemma 3.2, $|\mathcal{A}| \leq$

$|B \setminus X|l(\mathcal{F}, t+1) \leq s \cdot l(\mathcal{F}, t+1)$. Now let $C \in \mathcal{A}$. By Lemma 3.1, $|\mathcal{B}| \leq \binom{|C|}{t}l(\mathcal{G}, t) \leq \binom{r}{t}l(\mathcal{G}, t)$. Therefore,

$$|\mathcal{A}||\mathcal{B}| \leq s \cdot l(\mathcal{F}, t+1) \binom{r}{t} l(\mathcal{G}, t) \leq s \binom{r}{t} \frac{l(\mathcal{F}, t)}{c(r, s, t)} l(\mathcal{G}, t) < l(\mathcal{F}, t)l(\mathcal{G}, t).$$

Case 3: \mathcal{A} is not a trivial t -intersecting family, and \mathcal{B} is a trivial t -intersecting family. The result follows by an argument similar to that for Case 2.

Case 4: \mathcal{A} and \mathcal{B} are trivial t -intersecting families. Then $\mathcal{A} \subseteq \mathcal{F}(X)$ for some t -set X , and $\mathcal{B} \subseteq \mathcal{G}(Y)$ for some t -set Y . Thus $|\mathcal{A}| \leq l(\mathcal{F}, t)$, $|\mathcal{B}| \leq l(\mathcal{G}, t)$, and hence $|\mathcal{A}||\mathcal{B}| \leq l(\mathcal{F}, t)l(\mathcal{G}, t)$. Suppose $|\mathcal{A}||\mathcal{B}| = l(\mathcal{F}, t)l(\mathcal{G}, t)$. Then $|\mathcal{A}| = l(\mathcal{F}, t)$ and $|\mathcal{B}| = l(\mathcal{G}, t)$. Therefore, $\mathcal{A} = \mathcal{F}(X) \in \mathcal{L}(\mathcal{F}, t)$ and $\mathcal{B} = \mathcal{G}(Y) \in \mathcal{L}(\mathcal{G}, t)$.

Suppose $X \neq Y$. Then $X \setminus Y \neq \emptyset$ since $|X| = |Y| = t$. Let $x \in X \setminus Y$. Suppose $x \in B$ for all $B \in \mathcal{B}$. Then $\mathcal{B} \subseteq \mathcal{G}(Y \cup \{x\})$, and hence $|\mathcal{B}| \leq l(\mathcal{G}, t+1) \leq \frac{l(\mathcal{G}, t)}{c(r, s, t)} < l(\mathcal{G}, t)$, a contradiction. Thus $x \notin D$ for some $D \in \mathcal{B}$. Thus $X \not\subseteq D$. By Lemma 3.2, $|\mathcal{A}| \leq |D \setminus X|l(\mathcal{F}, t+1) \leq s \frac{l(\mathcal{F}, t)}{c(r, s, t)} < l(\mathcal{F}, t)$, a contradiction.

Therefore, $X = Y$. \square

Proof of Theorem 2.6. For $k = 2$, the result is given by Theorem 1.2. Consider $k \geq 3$. Let $(\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a cross- t -intersecting tuple below $(\mathcal{F}_1, \dots, \mathcal{F}_k)$. Then, for every $i, j \in [k]$ with $i \neq j$, \mathcal{A}_i and \mathcal{A}_j are cross- t -intersecting, and, since $r_1 \leq \dots \leq r_k$, we have $c(r_i, r_j, t) \leq c(r_{k-1}, r_k, t)$. By Theorem 1.2 and Lemma 2.5, $\prod_{i=1}^k |\mathcal{A}_i| \leq \prod_{i=1}^k l(\mathcal{F}_i, t)$. Suppose equality holds.

Suppose $|\mathcal{A}_h| < l(\mathcal{F}_h, t)$ for some $h \in [k]$. By Theorem 1.2 and Lemma 2.5, $\prod_{i \in [k] \setminus \{h\}} |\mathcal{A}_i| \leq \prod_{i \in [k] \setminus \{h\}} l(\mathcal{F}_i, t)$. Thus $\prod_{i=1}^k |\mathcal{A}_i| < \prod_{i=1}^k l(\mathcal{F}_i, t)$, a contradiction.

Therefore, $|\mathcal{A}_i| \geq l(\mathcal{F}_i, t)$ for each $i \in [k]$. Since $\prod_{i=1}^k |\mathcal{A}_i| = \prod_{i=1}^k l(\mathcal{F}_i, t)$, $|\mathcal{A}_i| = l(\mathcal{F}_i, t)$ for each $i \in [k]$. For each $i \in [2, k]$, we have $|\mathcal{A}_1||\mathcal{A}_i| = l(\mathcal{F}_1, t)l(\mathcal{F}_i, t)$, and hence, by Theorem 1.2, there exists a t -set $T_{1,i}$ such that $\mathcal{A}_1 = \mathcal{F}_1(T_{1,i}) \in \mathcal{L}(\mathcal{F}_1, t)$ and $\mathcal{A}_i = \mathcal{F}_i(T_{1,i}) \in \mathcal{L}(\mathcal{F}_i, t)$. By the argument in Case 4 of the proof of Theorem 1.2, $T_{1,i} = T_{1,2}$ for each $i \in [2, k]$. Thus $\mathcal{A}_i = \mathcal{F}_i(T_{1,2})$ for each $i \in [k]$. \square

4 Classes of families

In this section, we apply Theorem 2.6 to important classes of families. Thus, for each family \mathcal{F} , we need to obtain an upper bound for $\frac{l(\mathcal{F}, t)}{l(\mathcal{F}, t+1)}$ and compare it with $c(r, s, t)$.

Much of the work done on the t -intersection problem for the families treated here is outlined in [11]. Much less is known about the product cross- t -intersection problem because it takes the t -intersection problem to a deeper level; most of the main results are outlined in [15]. We will show that Theorem 2.6 provides a solution for many of the most natural and mostly studied classes of families. For each class, Theorem 1.4 provides a solution for the t -intersection problem.

4.1 Levels of power sets

For a family \mathcal{F} and a non-negative integer r , the family of all r -element sets in \mathcal{F} is called the r -th level of \mathcal{F} . For a set X , $\binom{X}{r}$ is the r -th level of 2^X .

Consider $\mathcal{F} = \binom{[n]}{p}$ with $1 \leq p \leq n$. Suppose $l(\mathcal{F}, t+1) > 0$ for some $t \geq 1$. Then $p \geq t+1$. We have

$$\frac{l(\mathcal{F}, t)}{l(\mathcal{F}, t+1)} = \frac{\binom{n-t}{p-t}}{\binom{n-t-1}{p-t-1}} = \frac{n-t}{p-t}. \quad (1)$$

Therefore, $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$ if $n \geq (p-t)c(r, s, t) + t$.

The following is a generalization of Theorem 1.1.

Theorem 4.1 *If $1 \leq t \leq r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, $\mathcal{F}_i = \binom{[n_i]}{r_i}$ with $n_i \geq (r_i - t)c(r_{k-1}, r_k, t) + t$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Proof. By (1), for each $i \in [k]$, $l(\mathcal{F}_i, t) \geq c(r_{k-1}, r_k, t)l(\mathcal{F}_i, t+1)$ as $n_i \geq (r_i - t)c(r_{k-1}, r_k, t) + t$. By Theorem 2.6, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property. Since $\mathcal{F}_i([t]) \in L(\mathcal{F}_i, t)$ for each $i \in [k]$, the result follows by Proposition 2.4. \square

4.2 Families of integer sequences

For an r -element set $X = \{x_1, \dots, x_r\}$ and an integer $m \geq 1$, we define

$$\mathcal{S}_{X,m} = \{ \{(x_1, y_1), \dots, (x_r, y_r)\} : y_1, \dots, y_r \in [m] \}.$$

Note that $\mathcal{S}_{X,m}$ is isomorphic to the set $[m]^r$, that is, the set of all sequences (y_1, \dots, y_r) such that $y_i \in [m]$ for each $i \in [r]$. We take $\mathcal{S}_{\emptyset, m}$ to be \emptyset . With a slight abuse of notation, for a family \mathcal{F} , we define

$$\mathcal{S}_{\mathcal{F}, m} = \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F, m}.$$

The t -intersection problem for $\mathcal{S}_{[n], m}$ was solved by Ahlswede and Khachatrian [2] and by Frankl and Tokushige [30], and that for $\mathcal{S}_{\mathcal{F}, m}$ is solved in [12] for m sufficiently large. The product cross- t -intersection problem for $\mathcal{S}_{[n], m}$ was solved by Moon [46] for $m \geq t+2$, and by Frankl et al. [29] and Pach and Tardos [47] for $m \geq t+1$. We solve the problem for $\mathcal{S}_{\mathcal{F}, m}$ with m sufficiently large depending only on t and the size of a largest set in \mathcal{F} .

Theorem 4.2 *If $1 \leq t \leq r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, \mathcal{F}_i is a $(\leq r_i)$ -family and $m_i \geq c(r_{k-1}, r_k, t)$, then $(\mathcal{S}_{\mathcal{F}_1, m_1}, \dots, \mathcal{S}_{\mathcal{F}_k, m_k})$ has the strict cross- t -star property.*

Lemma 4.3 *If $1 \leq t \leq r$ and \mathcal{F} is a $(\leq r)$ -family, then $l(\mathcal{S}_{\mathcal{F}, m}, t) \geq m \cdot l(\mathcal{S}_{\mathcal{F}, m}, t+1)$.*

Proof. Suppose $l(\mathcal{S}_{\mathcal{F},m}, t+1) > 0$. Then $r \geq t+1$. Let \mathcal{A} be a $(t+1)$ -star of $\mathcal{S}_{\mathcal{F},m}$ of size $l(\mathcal{S}_{\mathcal{F},m}, t+1)$. Then $\mathcal{A} = \mathcal{S}_{\mathcal{F},m}(Z)$ for some $(t+1)$ -element set Z . Let $\mathcal{G} = \{F \in \mathcal{F} : \mathcal{S}_{F,m}(Z) \neq \emptyset\}$. Let $T \in \binom{Z}{t}$. We have

$$\begin{aligned} l(\mathcal{S}_{\mathcal{F},m}, t+1) &= |\mathcal{S}_{\mathcal{F},m}(Z)| = \sum_{F \in \mathcal{F}} |\mathcal{S}_{F,m}(Z)| = \sum_{F \in \mathcal{G}} |\mathcal{S}_{F,m}(Z)| = \sum_{F \in \mathcal{G}} m^{|F|-t-1} \\ &= \frac{1}{m} \sum_{F \in \mathcal{G}} m^{|F|-t} = \frac{1}{m} \sum_{F \in \mathcal{G}} |\mathcal{S}_{F,m}(T)| \leq \frac{1}{m} \sum_{F \in \mathcal{F}} |\mathcal{S}_{F,m}(T)| \\ &= \frac{1}{m} |\mathcal{S}_{\mathcal{F},m}(T)| \leq \frac{1}{m} l(\mathcal{S}_{\mathcal{F},m}, t), \end{aligned}$$

and hence the result. \square

Proof of Theorem 4.2. For any $i \in [k]$, $l(\mathcal{S}_{\mathcal{F}_i, m_i}, t) \geq c(r_{k-1}, r_k, t) l(\mathcal{S}_{\mathcal{F}_i, m_i}, t+1)$ by Lemma 4.3 and the given condition $m_i \geq c(r_{k-1}, r_k, t)$. The result follows by Theorem 2.6. \square

Theorem 4.4 *If $1 \leq t \leq r$, \mathcal{F} is a $(\leq r)$ -family, $m \geq c(r, r, t)$, and $\mathcal{F}_1 = \dots = \mathcal{F}_k = \mathcal{S}_{\mathcal{F},m}$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Proof. The result follows by Theorem 4.2 and Proposition 2.3. \square

We make the following conjecture, which is analogous to [12, Conjecture 2.1].

Conjecture 4.5 *For any $t \geq 1$, there exists a positive integer $m_0(t)$ such that $(\mathcal{S}_{\mathcal{F},m}, \mathcal{S}_{\mathcal{F},m})$ has the strong cross- t -star property for any family \mathcal{F} and any $m \geq m_0(t)$.*

We also conjecture that the smallest possible $m_0(t)$ is $t+1$, and that $(\mathcal{S}_{\mathcal{F},m}, \mathcal{S}_{\mathcal{F},m})$ has the extrastrong cross- t -star property if $m > t+1$. By Lemma 2.5 and Proposition 2.3, this would imply a strengthening of Theorem 4.4. The conjecture does not hold for $m < t+1$. Indeed, it can be checked that, if $m \leq t$, $n \geq t+2$, $\mathcal{F} = \{[n]\}$, and $\mathcal{A} = \mathcal{B} = \{A \in \mathcal{S}_{[n],m} : |A \cap \{(1,1), \dots, (t+2,1)\}| \geq t+1\}$, then \mathcal{A} and \mathcal{B} are cross- t -intersecting subfamilies of $\mathcal{S}_{\mathcal{F},m}$ and $|\mathcal{A}||\mathcal{B}| > (m^{n-t})^2 = (l(\mathcal{S}_{\mathcal{F},m}))^2$.

4.3 Families of permutations

For an r -set $X = \{x_1, \dots, x_r\}$ and an integer $m \geq 1$, we define $\mathcal{S}_{X,m}^*$ to be the special subfamily of $\mathcal{S}_{X,m}$ given by

$$\mathcal{S}_{X,m}^* = \{\{(x_1, y_1), \dots, (x_r, y_r)\} : y_1, \dots, y_r \text{ are distinct elements of } [m]\}.$$

Note that $\mathcal{S}_{X,m}^* \neq \emptyset$ if and only if $r \leq m$. The family $\mathcal{S}_{X,m}^*$ can be interpreted as the set of permutations of sets in $\binom{[m]}{r}$; indeed, a member $\{(x_1, y_1), \dots, (x_r, y_r)\}$ of $\mathcal{S}_{X,m}^*$ corresponds uniquely to the permutation (y_1, \dots, y_r) of the r -element subset

$\{y_1, \dots, y_r\}$ of $[m]$. We take $\mathcal{S}_{\emptyset, m}$ to be \emptyset . With a slight abuse of notation, for a family \mathcal{F} , we define $\mathcal{S}_{\mathcal{F}, m}^*$ to be the special subfamily of $\mathcal{S}_{\mathcal{F}, m}$ given by

$$\mathcal{S}_{\mathcal{F}, k}^* = \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F, k}^*.$$

In [20], Deza and Frankl established the 1-star property of $\mathcal{S}_{[m], m}^*$ and conjectured that $\mathcal{S}_{[m], m}^*$ has the t -star property for m sufficiently large depending on t . Ellis, Friedgut and Pilpel [22] proved the conjecture together with the product cross- t -intersection version. The t -intersection problem for $\mathcal{S}_{\mathcal{F}, m}^*$ is solved in [12] for m sufficiently large depending only on t and the size of a largest set in \mathcal{F} . For this case, we have the following analogous result for the product cross- t -intersection problem.

Theorem 4.6 *If $1 \leq t \leq r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, \mathcal{F}_i is a $(\leq r_i)$ -family and $m_i \geq c(r_{k-1}, r_k, t) + t$, then $(\mathcal{S}_{\mathcal{F}_1, m_1}^*, \dots, \mathcal{S}_{\mathcal{F}_k, m_k}^*)$ has the strict cross- t -star property.*

Similarly to Theorem 4.2, this follows from the fact that if $\mathcal{S}_{F, m}^*(Z) \neq \emptyset$ for some $(t+1)$ -element Z , then $|\mathcal{S}_{F, m}^*(T)| = \frac{(m-t)!}{(m-|F|)!} = (m-t)|\mathcal{S}_{F, m}^*(Z)|$ for any $T \in \binom{Z}{t}$.

Theorem 4.7 *If $1 \leq t \leq r$, \mathcal{F} is a $(\leq r)$ -family, $m \geq c(r, r, t) + t$, and $\mathcal{F}_1 = \dots = \mathcal{F}_k = \mathcal{S}_{\mathcal{F}, m}^*$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Proof. The result follows by Theorem 4.6 and Proposition 2.3. \square

We make the following conjecture, which is analogous to [12, Conjecture 2.4].

Conjecture 4.8 *For any $t \geq 1$, there exists a positive integer $m_0^*(t)$ such that $(\mathcal{S}_{\mathcal{F}, m}^*, \mathcal{S}_{\mathcal{F}, m}^*)$ has the extrastrong cross- t -star property for any family \mathcal{F} and any $m \geq m_0^*(t)$.*

By Lemma 2.5 and Proposition 2.3, this would imply a strengthening of Theorem 4.7.

4.4 Families of multisets

A *multiset* is a collection A of objects such that each object possibly appears more than once in A . Thus the difference between a multiset and a set is that a multiset may have repetitions of its members. The *multiplicity* of a member a of a multiset A is the number of instances of a in A , and is denoted by $m_A(a)$. If a_1, \dots, a_r are the distinct members of a multiset A , then we can represent A uniquely by the set $\{(a_i, j) : i \in [r], j \in [m_A(a_i)]\}$, which we denote by S_A . Let $M_{n, r}$ denote the set of all multisets A such that the members of A are in $[n]$ and amount to r with repetitions included. An elementary counting result is that

$$|M_{n, r}| = \binom{n+r-1}{r}.$$

Let $\mathcal{M}_{n,r}$ denote the family $\{S_A: A \in M_{n,r}\}$. Note that two multisets A and B have exactly q common members (with repetitions included) if and only if $|S_A \cap S_B| = q$.

The t -intersection problem for $\mathcal{M}_{n,r}$ was solved by Meagher and Purdy [45] for $t = 1$, and by Füredi, Gerbner and Vizer [31] for $n \geq 2r - t$. Here we solve the product cross- t -intersection problem for n sufficiently large depending on r and t .

Consider $\mathcal{F} = \mathcal{M}_{n,p}$. Suppose $l(\mathcal{F}, t+1) > 0$ for some $t \geq 1$. Then $p \geq t+1$. We have

$$\frac{l(\mathcal{F}, t)}{l(\mathcal{F}, t+1)} = \frac{\binom{n+p-t-1}{p-t}}{\binom{n+p-t-2}{p-t-1}} = \frac{n+p-t-1}{p-t}. \quad (2)$$

Therefore, $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$ if $n \geq (p-t)c(r, s, t) - p + t + 1$.

Theorem 4.9 *If $1 \leq t \leq r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, $\mathcal{F}_i = \mathcal{M}_{n_i, r_i}$ with $n_i \geq (r_i - t)c(r_{k-1}, r_k, t) - r_i + t + 1$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Proof. For each $i \in [k]$, $l(\mathcal{F}_i, t) \geq c(r_{k-1}, r_k, t)l(\mathcal{F}_i, t+1)$ by (2) and the given condition $n_i \geq (r_i - t)c(r_{k-1}, r_k, t) - r_i + t + 1$. By Theorem 2.6, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property. Let $T = \{(1, i): i \in [t]\}$. Since $\mathcal{F}_i(T) \in L(\mathcal{F}_i, t)$ for each $i \in [k]$, the result follows by Proposition 2.4. \square

4.5 Families of compositions

If a_1, a_2, \dots, a_r and n are positive integers such that $n = a_1 + a_2 + \dots + a_r$, then the tuple (a_1, a_2, \dots, a_r) is said to be a *composition of n of length r* . Let $C_{n,r}$ denote the set of all compositions of n of length r . An elementary counting result is that

$$|C_{n,r}| = \binom{n-1}{n-r} = \binom{n-1}{r-1}.$$

We can represent a composition $\mathbf{a} = (a_1, \dots, a_r)$ uniquely by the set $\{(1, a_1), \dots, (r, a_r)\}$, which we denote by $S_{\mathbf{a}}$. Let $\mathcal{C}_{n,r}$ denote the family $\{S_{\mathbf{a}}: \mathbf{a} \in C_{n,r}\}$.

We say that a composition $\mathbf{a} = (a_1, \dots, a_r)$ *strongly t -intersects* a composition $\mathbf{b} = (b_1, \dots, b_s)$ if there exists a t -element subset T of $[\min\{r, s\}]$ such that $a_i = b_i$ for each $i \in T$. Note that \mathbf{a} *strongly t -intersects* \mathbf{b} if and only if $|S_{\mathbf{a}} \cap S_{\mathbf{b}}| \geq t$.

Ku and Wong [41] solved the t -intersection problem for $\mathcal{C}_{n,r}$ with n sufficiently large. In [42], they also proved Theorem 4.10 below for sufficiently large values of n_1, \dots, n_r .

Consider $\mathcal{F} = \mathcal{C}_{n,p}$ with $t+1 < p \leq n$. It is straightforward that $\mathcal{F}(\{(i, 1): i \in [t]\})$ is a largest t -star of $\mathcal{C}_{n,p}$. We have

$$\frac{l(\mathcal{F}, t)}{l(\mathcal{F}, t+1)} = \frac{\binom{n-t-1}{p-t-1}}{\binom{n-t-2}{p-t-2}} = \frac{n-t-1}{p-t-1}. \quad (3)$$

Therefore, $l(\mathcal{F}, t) \geq c(r, s, t)l(\mathcal{F}, t+1)$ if $n \geq (p-t-1)c(r, s, t) + t + 1$.

Theorem 4.10 *If $2 \leq t+1 < r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, $\mathcal{F}_i = \mathcal{C}_{n_i, r_i}$ with $n_i \geq (r_i - t - 1)c(r_{k-1}, r_k, t) + t + 1$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Proof. For each $i \in [k]$, $l(\mathcal{F}_i, t) \geq c(r_{k-1}, r_k, t)l(\mathcal{F}_i, t+1)$ by (3) and the given condition $n_i \geq (r_i - t - 1)c(r_{k-1}, r_k, t) + t + 1$. By Theorem 2.6, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property. Let $T = \{(i, 1) : i \in [t]\}$. Since $\mathcal{F}_i(T) \in L(\mathcal{F}_i, t)$ for each $i \in [k]$, the result follows by Proposition 2.4. \square

4.6 Families of set partitions

If X_1, X_2, \dots, X_r are pairwise disjoint non-empty sets and $X = \bigcup_{i=1}^r X_i$, then the set $\{X_1, X_2, \dots, X_r\}$ is called a *partition of X of length r* , and X_1, X_2, \dots, X_r are called the *parts* of the partition. Let $\mathbf{P}_{n,r}$ denote the family of all partitions of $[n]$ of length r , and let $s_{n,r} = |\mathbf{P}_{n,r}|$. Trivially, $s_{n,1} = 1 = s_{n,n}$. An elementary result is that

$$s_{n,r} = s_{n-1,r-1} + r s_{n-1,r} \quad \text{if } 2 \leq r \leq n-1.$$

It follows that

$$s_{m,r} \leq s_{n,r} \quad \text{if } 1 \leq m \leq n. \quad (4)$$

Lemma 4.11 *If $1 < r < n$, then $s_{n,r} \geq \frac{n-1}{r-1} s_{n-1,r-1}$.*

Proof. Consider any $X \in \mathbf{P}_{n-1,r-1}$. For any $i \in [n-1]$, let A_i be the part of X that contains i , and let X_i be the member of $\mathbf{P}_{n,r}$ obtained by replacing i by n in A_i , and adding $\{i\}$ as a part; that is, $X_i = (X \setminus \{A_i\}) \cup \{(A_i \setminus \{i\}) \cup \{n\}\} \cup \{\{i\}\}$. For any $Y \in \mathbf{P}_{n,r}$, let $f(X_i, Y) = 1$ if $X_i = Y$, and let $f(X_i, Y) = 0$ if $X_i \neq Y$. If Y has no parts of size 1, then $f(X_i, Y) = 0$. Suppose that $\{y_1\}, \dots, \{y_p\}$ are the distinct parts of Y of size 1. Since $n > r$, $p \leq r-1$. Let B be the part of Y that contains n . Then $f(X_i, Y) = 1$ if and only if $i \in \{y_1, \dots, y_p\}$ and $X = (Y \setminus \{B, \{i\}\}) \cup \{(B \setminus \{n\}) \cup \{i\}\}$.

Therefore, we have

$$\begin{aligned} (n-1)s_{n-1,r-1} &= \sum_{X \in \mathbf{P}_{n-1,r-1}} \sum_{i=1}^{n-1} 1 = \sum_{X \in \mathbf{P}_{n-1,r-1}} \sum_{i=1}^{n-1} \sum_{Y \in \mathbf{P}_{n,r}} f(X_i, Y) \\ &= \sum_{Y \in \mathbf{P}_{n,r}} \sum_{X \in \mathbf{P}_{n-1,r-1}} \sum_{i=1}^{n-1} f(X_i, Y) \leq \sum_{Y \in \mathbf{P}_{n,r}} (r-1) = (r-1)s_{n,r}, \end{aligned}$$

and hence the result. \square

Erdős and Székely [24] solved the t -intersection problem for $\mathbf{P}_{n,r}$ with n sufficiently large (see [40] for a related result). Using the results above, we prove the following cross- t -intersection result.

Theorem 4.12 *If $2 \leq t + 1 < r_1 \leq \dots \leq r_k$ and, for each $i \in [k]$, $\mathcal{F}_i = \mathcal{P}_{n_i, r_i}$ with $n_i \geq (r_i - t - 1)c(r_{k-1}, r_k, t) + t + 1$, then $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the extrastrong cross- t -star property.*

Lemma 4.13 *If $1 \leq t < r \leq n$, then $l(\mathcal{P}_{n,r}, t) = s_{n-t, r-t}$.*

Proof. Let $T = \{\{i\} : i \in [t]\}$. We have $l(\mathcal{P}_{n,r}, t) \geq |\mathcal{P}_{n,r}(T)| = s_{n-t, r-t}$. Let \mathcal{A} be a largest t -star of $\mathcal{P}_{n,r}$. There exist t pairwise disjoint non-empty subsets X_1, \dots, X_t of $[n]$ such that $\mathcal{A} = \mathcal{P}_{n,r}(\{X_1, \dots, X_t\})$. Thus $l(\mathcal{P}_{n,r}, t) = |\mathcal{A}| = s_{n', r-t}$, where $n' = n - \sum_{i=1}^t |X_i| \leq n - t$. By (4), $l(\mathcal{P}_{n,r}, t) \leq s_{n-t, r-t}$. Since $l(\mathcal{P}_{n,r}, t) \geq s_{n-t, r-t}$, the result follows. \square

Lemma 4.14 *If $2 \leq t + 1 < r < n$, then $l(\mathcal{P}_{n,r}, t) \geq \frac{n-t-1}{r-t-1}l(\mathcal{P}_{n,r}, t+1)$.*

Proof. By Lemma 4.13, $l(\mathcal{P}_{n,r}, t) = s_{n-t, r-t}$ and $l(\mathcal{P}_{n,r}, t+1) = s_{n-t-1, r-t-1}$. Thus, by Lemma 4.11, $l(\mathcal{P}_{n,r}, t) \geq \frac{n-t-1}{r-t-1}l(\mathcal{P}_{n,r}, t+1)$. \square

Proof of Theorem 4.12. For each $i \in [k]$, $l(\mathcal{F}_i, t) \geq c(r_{k-1}, r_k, t)l(\mathcal{F}_i, t+1)$ by Lemma 4.14 and the given condition $n_i \geq (r_i - t - 1)c(r_{k-1}, r_k, t) + t + 1$. By Theorem 2.6, $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ has the strict cross- t -star property. Let $T = \{\{i\} : i \in [t]\}$. For each $i \in [k]$, we have $|\mathcal{F}_i(T)| = s_{n_i-t, r_i-t}$, and hence $\mathcal{F}_i(T) \in \mathcal{L}(\mathcal{F}_i, t)$ by Lemma 4.13. The result follows by Proposition 2.4. \square

References

- [1] R. Ahlswede and L.H. Khachatrian, The complete intersection theorem for systems of finite sets, *European J. Combin.* 18 (1997), 125–136.
- [2] R. Ahlswede and L.H. Khachatrian, The diametric theorem in Hamming spaces—Optimal anti-codes, *Adv. Appl. Math.* 20 (1998), 429–449.
- [3] M.O. Albertson and K.L. Collins, Homomorphisms of 3-chromatic graphs, *Discrete Math.* 54 (1985), 127–132.
- [4] C. Bey, On cross-intersecting families of sets, *Graphs Combin.* 21 (2005), 161–168.
- [5] P. Borg, A cross-intersection theorem for subsets of a set, *Bull. London. Math. Soc.* 47 (2015), 248–256.
- [6] P. Borg, A short proof of a cross-intersection theorem of Hilton, *Discrete Math.* 309 (2009), 4750–4753.
- [7] P. Borg, Cross-intersecting families of permutations, *J. Combin. Theory Ser. A* 117 (2010), 483–487.
- [8] P. Borg, Cross-intersecting families of partial permutations, *SIAM J. Disc. Math.* 24 (2010), 600–608.
- [9] P. Borg, Cross-intersecting sub-families of hereditary families, *J. Combin. Theory Ser. A* 119 (2012), 871–881.
- [10] P. Borg, Extremal t -intersecting sub-families of hereditary families, *J. London Math. Soc.* 79 (2009), 167–185.

- [11] P. Borg, Intersecting families of sets and permutations: a survey, in: *Advances in Mathematics Research* (A.R. Baswell Ed.), Volume 16, Nova Science Publishers, Inc., 2011, pp. 283–299.
- [12] P. Borg, On t -intersecting families of signed sets and permutations, *Discrete Math.* 309 (2009), 3310–3317.
- [13] P. Borg, The maximum product of sizes of cross- t -intersecting uniform families, *Australas. J. Combin.* 60 (2014), 69–78.
- [14] P. Borg, The maximum product of weights of cross-intersecting families, [arXiv:1512.09108 \[math.CO\]](https://arxiv.org/abs/1512.09108).
- [15] P. Borg, The maximum sum and the maximum product of sizes of cross-intersecting families, *European J. Combin.* 35 (2014), 117–130.
- [16] P. Borg and I. Leader, Multiple cross-intersecting families of signed sets, *J. Combin. Theory Ser. A* 117 (2010), 583–588.
- [17] P.J. Cameron and C.Y. Ku, Intersecting families of permutations, *European J. Combin.* 24 (2003), 881–890.
- [18] V. Chvátal, Unsolved Problem No. 7, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), *Hypergraph Seminar, Lecture Notes in Mathematics*, Vol. 411, Springer, Berlin, 1974.
- [19] D.E. Daykin, Erdős-Ko-Rado from Kruskal-Katona, *J. Combin. Theory Ser. A*, 17(1974), pp. 254–255.
- [20] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *J. Combin. Theory Ser. A* 22 (1977) 352–360.
- [21] M. Deza and P. Frankl, The Erdős-Ko-Rado theorem—22 years later, *SIAM J. Algebraic Discrete Methods* 4 (1983), pp. 419–431.
- [22] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. Amer. Math. Soc.* 24 (2011), 649–682.
- [23] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* (2) 12 (1961), 313–320.
- [24] P.L. Erdős and L.A. Székely, Erdős-Ko-Rado theorems of higher order, in: I. Althöfer, Ning Cai, G. Dueck, L. Khachatryan, M.S. Pinski, A. Sarközy, I. Wegener, Zhen Zhang (Eds.), *Numbers, Information and Complexity*, Kluwer Academic, 2000, pp. 117–124.
- [25] P. Frankl, Extremal set systems, in: R.L. Graham, M. Grötschel and L. Lovász (Eds.), *Handbook of Combinatorics*, Vol. 2, Elsevier, Amsterdam, 1995, pp. 1293–1329.
- [26] P. Frankl, The Erdős-Ko-Rado Theorem is true for $n = ckt$, *Proc. Fifth Hung. Comb. Coll.*, North-Holland, Amsterdam, 1978, pp. 365–375.
- [27] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), *Surveys in Combinatorics*, Cambridge Univ. Press, London/New York, 1987, pp. 81–110.
- [28] P. Frankl and Z. Füredi, Beyond the Erdős-Ko-Rado theorem, *J. Combin. Theory Ser. A* 56 (1991), 182–194.
- [29] P. Frankl, S.J. Lee, M. Siggers and N. Tokushige, An Erdős-Ko-Rado theorem for cross t -intersecting families, *J. Combin. Theory Ser. A* 128 (2014), 207–249.
- [30] P. Frankl and N. Tokushige, The Erdős-Ko-Rado theorem for integer sequences, *Combinatorica* 19 (1999), 55–63.
- [31] Z. Füredi, D. Gerbner and M. Vizer, A discrete isodiametric result: the Erdős-Ko-Rado theorem for multisets, *European J. Combin.* 48 (2015), 224–233.

- [32] A.J.W. Hilton, An intersection theorem for a collection of families of subsets of a finite set, J. London Math. Soc. (2) 15 (1977), 369–376.
- [33] J. Hirschorn, Asymptotic upper bounds on the shades of t -intersecting families, arXiv:0808.1434.
- [34] F.C. Holroyd, C. Spencer and J. Talbot, Compression and Erdős-Ko-Rado graphs, Discrete Math. 293 (2005) 155–164.
- [35] F.C. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, Discrete Math. 293 (2005) 165–176.
- [36] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, J. Combin. Theory Ser. B 13 (1972), pp. 183–184.
- [37] G.O.H. Katona, A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, Akadémiai Kiadó, 1968, pp. 187–207.
- [38] G.O.H. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hungar. 15 (1964) 329–337.
- [39] J.B. Kruskal, The number of simplices in a complex, in: Mathematical Optimization Techniques, University of California Press, Berkeley, California, 1963, pp. 251–278.
- [40] C.Y. Ku and D. Renshaw, Erdős-Ko-Rado theorems for permutations and set partitions, J. Combin. Theory Ser. A 115 (2008), 1008–1020.
- [41] C.Y. Ku and K.B. Wong, An analogue of the Erdős-Ko-Rado theorem for weak compositions, Discrete Math. 313 (2013), 2463–2468.
- [42] C.Y. Ku and K.B. Wong, On r -cross t -intersecting families for weak compositions, Discrete Math. 338 (2015), 1090–1095.
- [43] M. Matsumoto and N. Tokushige, A generalization of the Katona theorem for cross t -intersecting families, Graphs Combin. 5 (1989), 159–171.
- [44] M. Matsumoto and N. Tokushige, The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families, J. Combin. Theory Ser. A 52 (1989), 90–97.
- [45] K. Meagher and A. Purdy, An Erdős-Ko-Rado theorem for multisets, Electron. J. Combin. 18(1) (2011), P220.
- [46] A. Moon, An analogue of the Erdős-Ko-Rado theorem for the Hamming schemes $H(n, q)$, J. Combin. Theory Ser. A 32 (1982) 386–390.
- [47] J. Pach and G. Tardos, Cross-intersecting families of vectors, Graphs Combin. 31 (2015), 477–495.
- [48] L. Pyber, A new generalization of the Erdős-Ko-Rado theorem, J. Combin. Theory Ser. A 43 (1986), 85–90.
- [49] N. Tokushige, On cross t -intersecting families of sets, J. Combin. Theory Ser. A 117 (2010), 1167–1177.
- [50] N. Tokushige, The eigenvalue method for cross t -intersecting families, J. Alg. Comb. 38 (2013), 653–662.
- [51] J. Wang and H. Zhang, Cross-intersecting families and primitivity of symmetric systems, J. Combin. Theory Ser. A 118 (2011), 455–462.
- [52] R.M. Wilson, The exact bound in the Erdős-Ko-Rado theorem, Combinatorica 4 (1984) 247–257.